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Statistics of coincidences

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1. Event statistics in a single detector

Consider a particle detector counting random events with a duration τ which is very small compared to the average time T between events: $\tau \ll T$. More precisely, we assume that the chance of overlapping events in time T is of order $(\tau/T)^2$ or less. Then the probability of observing n events per unit of time is given by the Poisson distribution

$$P_N(n) = \frac{N^n}{n!}\, e^{-N}, \qquad \sum_{n=0}^{\infty} P_N(n) = 1, \tag{1}$$

where N is the average (expected) number of events per unit of time. Check:

$$\langle n \rangle = \sum_{n=1}^{\infty} n P_N(n) = \sum_{n=0}^{\infty} \frac{N^{n+1}}{n!}\, e^{-N} = N. \tag{2}$$

Similarly the fluctuations can be calculated:

$$\langle (n-N)^2 \rangle = \langle n(n-1) \rangle + N - N^2,$$

and

$$\langle n(n-1) \rangle = \sum_{n=2}^{\infty} n(n-1) P_N(n) = \sum_{n=0}^{\infty} \frac{N^{n+2}}{n!}\, e^{-N} = N^2.$$

Combining these results, we find for the average size of a fluctuation

$$\sqrt{\langle (n-N)^2 \rangle} = \sqrt{N}. \tag{3}$$

If we take as unit of time the second, then N is the number of random events per second, and $\sqrt{N}$ the mean square root deviation from the average per second. Even for a small expectation value like $N = 4$ the fraction of samples of size n with values in the interval $[N-\sqrt{N}, N+\sqrt{N}]$ (i.e. $2 \leq n \leq 6$) is 57 %. For large N this approaches the gaussian value 68.3 %. For $N = 9$ the number of samples of size n with values in the interval $[N - 2\sqrt{N}, N + 2\sqrt{N}]$ is 92 %, close to the gaussian value 95.5 % which is approached for large N.

2. Coincidences between 2 detectors

We now consider two detectors, characterized by an average count of N_1 events with duration τ_1 in detector 1, and N_2 events with duration τ_2 in detector 2. If the events are of the same type, and if there is no dead time (or if the dead times of each detector are equal), then of course $\tau_1 = \tau_2$. Now suppose that each detector only observes random (background) events, i.e. there is no source of detectable signals. In a unit of time detector 1 will detect n_1 events with probability $P_{N_1}(n_1)$, and detector 2 will detect n_2 events with probability $P_{N_2}(n_2)$.

Now we ask what is the chance to observe coincidences between the two detectors. We first have to define what we mean by a coincidence. In this section we mean by a coincidence the occurrence of one event in both detectors within a time window $\tau_1 + \tau_2$. Now if there are n_1 events in detector 1, then the total time window for overlapping events in detector 2 is $n_1(\tau_1 + \tau_2)$; given there are n_2 events observed per unit of time in detector 2, we expect

$$c(1,2) = n_1 n_2 (\tau_1 + \tau_2) \tag{4}$$

overlapping events to be observed per unit of time in detector 2. This is then the number of coincidences in unit time. Since the detectors are supposed to be independent, the probability for observing n_1 events per unit of time in detector 1 and n_2 events per unit time in detector 2 is given by the product of the single-detector probabilities. Hence the total probability for oberving $c(1,2)$ coincidences in detectors with average rates N_1 and N_2 is

$$W_{(N_1,N_2)}(1,2) = (\tau_1 + \tau_2)\, n_1 n_2\, P_{N_1}(n_1) P_{N_2}(n_2). \tag{5}$$

By the independence of the detectors, the *expected* number of coincidences per unit of time is then

$$\langle c(1,2)\rangle = N_1 N_2\, (\tau_1 + \tau_2). \tag{6}$$

We can also calculate the expected fluctuations:

$$\begin{aligned} \langle (n_1 n_2 - N_1 N_2)^2 \rangle (\tau_1 + \tau_2)^2 &= \left(\langle n_1^2 \rangle \langle n_2^2 \rangle - N_1^2 N_2^2 \right) (\tau_1 + \tau_2)^2 \\ &= N_1 N_2 (N_1 + N_2 + 1)(\tau_1 + \tau_2)^2. \end{aligned} \tag{7}$$

Hence the m.r.s. deviations are

$$\sqrt{\langle (c(1,2) - \langle c(1,2)\rangle)^2 \rangle} = \sqrt{N_1 N_2 (N_1 + N_2 + 1)}\, (\tau_1 + \tau_2)\,, \tag{8}$$

and the relative m.r.s. fluctuations

$$\sqrt{\frac{\langle (c(1,2) - \langle c(1,2)\rangle)^2 \rangle}{\langle c(1,2)\rangle^2}} = \sqrt{\frac{N_1 + N_2 + 1}{N_1 N_2}}. \tag{9}$$

In particular, for $N_1 = N_2 \gg 1$ this becomes $\sqrt{2/N}$. Observe, that this quantity does not depend on the duration of the events τ_i.

3. Coincidences between 3 detectors

The previous analysis can easily be extended to coincidences between 3 or more detectors. Of course, we must again define precisely what we mean by a triple coincidence. The straightforward generalization of the previous case is to define a coincidence as any set of events in each of the three detectors within a time interval $\Delta\tau = (\tau_1 + \tau_2 + \tau_3)$. Of course, this is not the only possibility. We could also require a coincidence within a time window $\Delta\tau = \tau_s + \tau_l$, the combined shortest and longest among the three event times. This latter definition has the advantage that it can be taken over unchanged for the definition of n-fold coincidences.

If n_i is the number of events in detector i in the unit of time in which the triple coincidence occurs, then the total time occupied by events in detector 1 is $n_1 \Delta\tau$. During this time we expect a number of events in detector 2

$$c_2(1,2) = n_1 n_2 \Delta\tau. \tag{10}$$

Then the total time window occupied by double coincidences in detectors 1 and 2 is

$$c_2(1,2)\Delta\tau = n_1 n_2 (\Delta\tau)^2. \tag{11}$$

In this window there are

$$c_3(1,2,3) = n_1 n_2 n_3 \, (\Delta\tau)^2 \tag{12}$$

events in detector 3. This represents the number of triple coincidences in the unit of time under consideration. As the probabilities for observing n_i events in detector i in a given unit time are independent, the probability for the triple coincidence (**??**) is

$$W_{N_1,N_2,N_3}(1,2,3) = (\Delta\tau)^2 \, n_1 n_2 n_3 \, P_{N_1}(n_1) P_{N_2}(n_2) P_{N_3}(n_3), \tag{13}$$

and the *expected* number of triple events per unit of time is

$$\langle c_3(1,2,3) \rangle = N_1 N_2 N_3 \, (\Delta\tau)^2. \tag{14}$$

The fluctuations around this expectation value are

$$\begin{aligned} \langle (c_3(1,2,3) - \langle c_3(1,2,3) \rangle)^2 \rangle &= \left(\langle n_1^2 \rangle \langle n_2^2 \rangle \langle n_3^2 \rangle - N_1^2 N_2^2 N_3^2 \right) (\Delta\tau)^4 \\ &= N_1 N_2 N_3 \, (N_1 N_2 + N_2 N_3 + N_3 N_1 + N_1 + N_2 + N_3 + 1) \, (\Delta\tau)^4. \end{aligned} \tag{15}$$

Then the relative r.m.s. fluctuations are (in the limit of large N_i)

$$\sqrt{\frac{\langle (c_3(1,2,3) - \langle c_3(1,2,3) \rangle)^2 \rangle}{\langle c_3(1,2,3) \rangle^2}} = \sqrt{\frac{N_1 N_2 + N_2 N_3 + N_3 N_1}{N_1 N_2 N_3}}. \tag{16}$$

If all N_i are equal, this reduces to $\sqrt{3/N}$. The systematics is obvious.